

David Vogan 1 PM

Affine Weyl Groups

$$\begin{array}{ll} X^* & X \\ U & = \\ R & U \\ U & R^\vee \\ R^+ & U \\ & \vdots \\ U & \end{array}$$

II - simple

next Datum $\rightsquigarrow W = \mathbb{Z}^-$ linear automorphisms of X^*
 that is generated by $\{s_\alpha : \alpha \in \Pi\}$
 or $\{s_\beta : \beta \in R\}$
 $s_\alpha = 1 - \langle \alpha, \alpha^\vee \rangle \alpha$

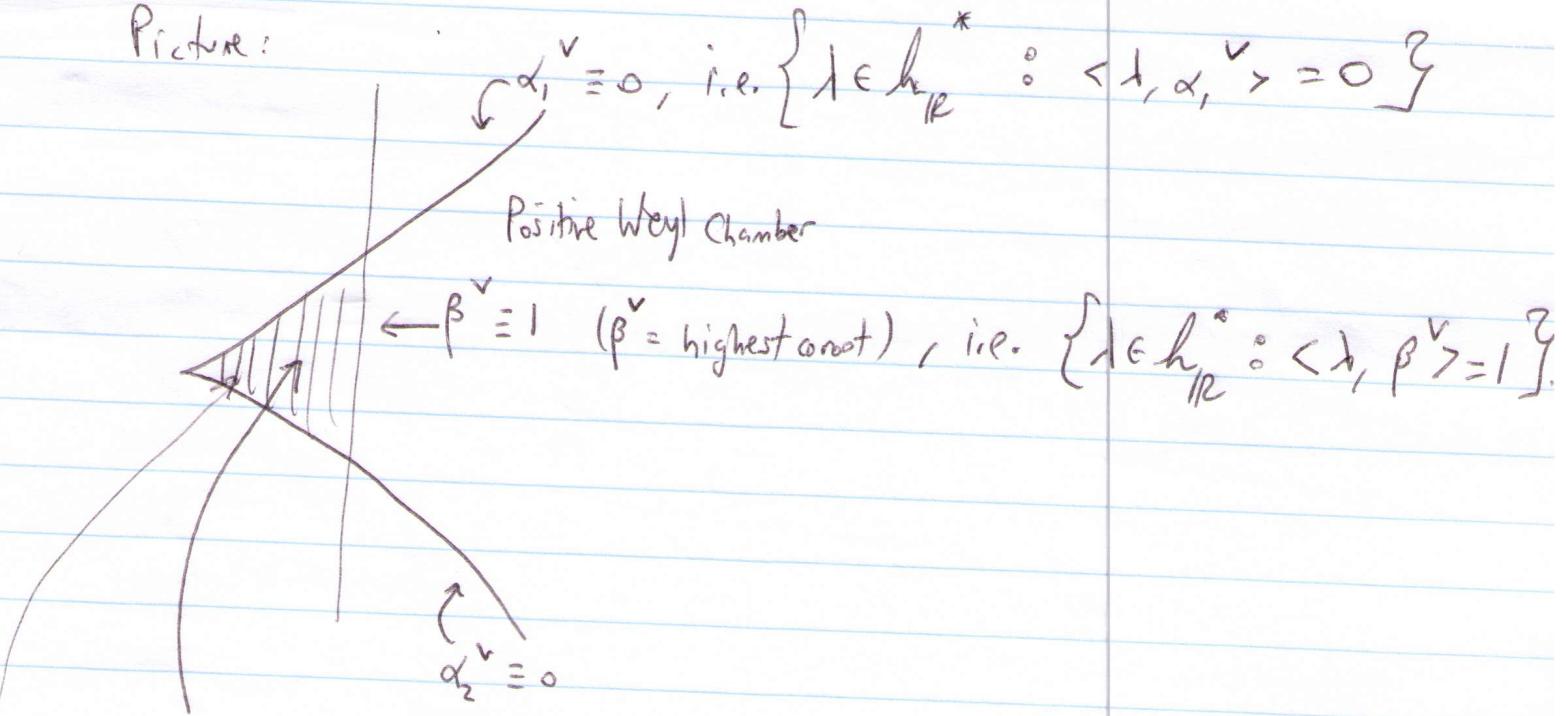
$V^\alpha = \text{Affine Weyl Group} = \mathbb{Z}\text{-affine transformations of } X^*$ generated by
 the affine reflections $s_\beta = 1 - \langle \beta, \beta^\vee \rangle \beta$,
 the reflection in the hyperplane $\langle \beta, \beta^\vee \rangle = n$.
 where $\beta \in R$, $n \in \mathbb{Z}$ (with this defn, this is related to dual group, not the group)

$W^{\text{aff}} = W \ltimes (\text{Root Lattice})$, where Root Lattice
 is acting by translation.

W^{aff} is a Coxeter group, one for each simple factor of R .

$\{s_\alpha : \alpha \in \Pi\} \cup \{s_{\beta_i} : \beta_i \text{ a highest root}\}$

Picture:



fundamental domain for W^{aff} action on $h_{\mathbb{R}}^* = X^* \otimes_{\mathbb{Z}} \mathbb{R}$

$W^{\text{ext}} := W \backslash X^* \supset W^{\text{aff}}$, $W^{\text{ext}} / W^{\text{aff}} = X^* / \mathbb{R} \cong$ algebraic characters of $\mathbb{Z}(G)$

$$h_{\mathbb{R}}^* := X^* \otimes_{\mathbb{Z}} \mathbb{R}.$$

$$h_{\mathbb{R}}^* / W^{\text{ext}} \cong {}^v G - \text{conjugacy classes of semisimple, elliptic elements of } {}^v G$$

= Conjugacy classes in compact form of ${}^v G$

✓ easy, David said.

elliptic means all eigenvalues are on the unit circle

If X^* = root lattice, so G is adjoint, ${}^v G$ r.c., then

$$h_{\mathbb{R}}^* / W^{\text{ext}} \cong h^* / W^{\text{aff}} \cong \text{fundamental domain}$$

~~$= \{ \lambda \in h_{\mathbb{R}}^* : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \forall i \in I \}$~~

$= \{ \lambda \in h_{\mathbb{R}}^*: \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ s.t. } \alpha_i \text{ simple}$

and $\langle \lambda, \rho^\vee \rangle \leq 1$, ρ^\vee any highest root $\}$

Example: ~~Let~~ let α be a simple root. Then

Coherent Families of representations

let $h_{\mathbb{C}}^* = X^* \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\boxed{\frac{h_{\mathbb{C}}^*}{W^{\text{ext}}} \cong {}^G \text{conjugacy classes of semisimple elements}}$

$\boxed{h_{\mathbb{C}}^*/W \cong \text{homomorphisms } \beta(g) \rightarrow \mathbb{C}}$

↑
Harish-Chandra
homomorphism

Fix $\lambda_0 \in h_{\mathbb{C}}^*$. Coherent Family based on $\lambda_0 + X^*$ is

symbol or subset of $h_{\mathbb{C}}^*$

a map

$(H): \lambda_0 + X^* \longrightarrow \text{virtual reprs of } G_{\mathbb{R}}$ of finite length

such that

① $\textcircled{H}(\lambda_0 + \lambda)$ has inf'l character $\lambda_0 + \lambda \in h^*_\mathbb{C}$

② If F is a finite dim'l algebraic repn of $G_\mathbb{C}$,

then
$$F \otimes \textcircled{H}(\lambda_0 + \lambda) = \sum_{\mu \text{ a weight of } F} \textcircled{H}(\lambda_0 + \lambda + \mu) \cdot m_{\mu, F}$$

where $m_{\mu, F}$ = multiplicity of μ as a weight of F .

One goal: Make an action of W^{aff} on stuff like this
may be an action on the set of coherent families

Well, try to make W act to get a new coherent family.

Try $(w \cdot \textcircled{H})(\lambda_0 + \lambda) := \textcircled{H}(w\lambda_0 + w\lambda)$

~~Then~~ Then $w \cdot \textcircled{H}$ is a coherent family based on $w\lambda_0 + \lambda$ *

So you get a W -action on $\text{span}\{\text{coherent families on } w\lambda_0 + \lambda\}$ various

Denote $W_{[1]} = \{w \in W : w^{-1}\lambda_0 - \lambda_0 \in X^*\}$.

You get a rep'n of $W_{[\lambda_0]}$ on

$$\mathbb{Z}\text{-span}\{\text{coherent families on } \lambda_0 + X^*\}$$

{finite rank \mathbb{Z} -module}

Also get $\text{Ind}_{W_{[\lambda_0]}}^W$ (this last rep'n).

This last thing is boring.

$$W_{[\lambda_0]} \supset W_{\lambda_0} := \{ w \in W : w^{-1}\lambda_0 - \lambda_0 \in \text{root lattice} \}.$$

= parabolic subgroup of W^{aff}

λ_0 semisimple
min $e(\lambda_0)$ = ~~elliptic~~ element in ${}^r G \supset {}^r H$

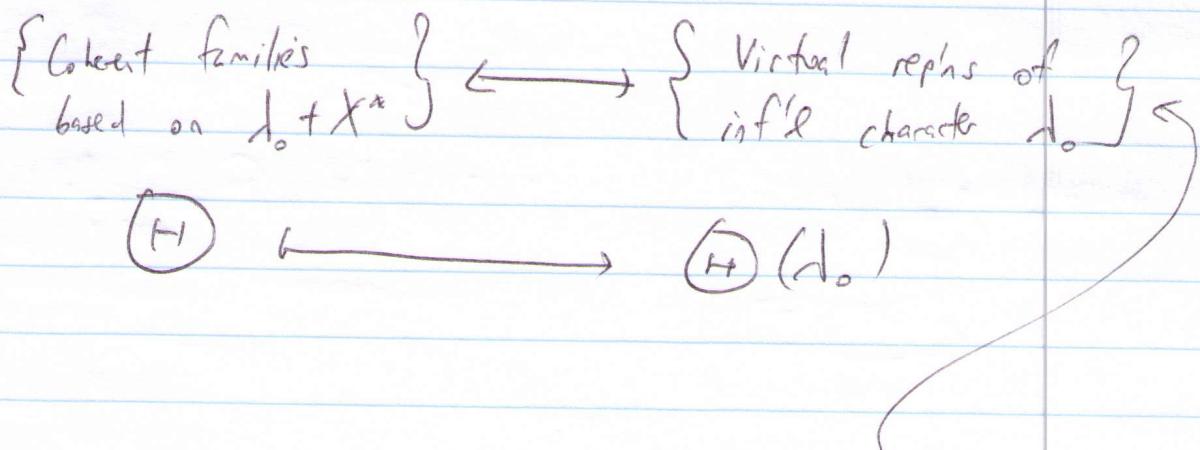
(iii) centralizer of $e(\lambda_0)$ in ${}^r G$,

↳ reductive maybe disconnected

$$\cancel{{}^r G} \quad W(\text{Cent}(e(\lambda_0)), {}^r H) \cong W_{[\lambda_0]}$$

$$W_{[\lambda_0]} / W_{\lambda_0} \cong \text{Cent}_{{}^r G}(e(\lambda_0)) / \text{Cent}_{{}^r G}(e(\lambda_0))^0 \leftarrow \begin{matrix} \text{identity} \\ \text{component} \end{matrix}$$

If d_0 is regular, then the ~~Coleant families~~



basis is the irred. w/ infl character d_0

Fix ^a Coleant family on $d_0 + X^*$, and regular d_0 .

$$H(d_0) = \sum m_i X_i, \quad m_i \in \mathbb{Z}$$

X_i irred.

X_i associated variety = union of closures of nilpotent $K_{\mathbb{C}}^*$ -orbits on

$$\mathcal{O}/k \quad (\mathcal{O}/k)^*$$

List the maximal orbits as X_i varieties: $\Omega_1, \dots, \Omega_m$.

↳ independent of choice of d_0 .

Each \mathcal{O}_j is $\cong K_C / \mathbb{C}^{\times}$

where $\xi_j \in \mathbb{C}^{\times} (\mathcal{O}_j / K_C)$ is an orbit representative.

There's an associated cycle construction which

associates to each x_i a ^{new} virtual

algebraic rep'n of

$K_C^{\xi_j}$, call it $\tilde{\epsilon}_{ij}(\lambda_0)$.

\sum means a sum of words with non-negative coefficients.

define $\tilde{\epsilon}_j(\lambda_0) := \sum_i m_i \tilde{\epsilon}_{ij}(\lambda_0)$, a virtual repn.

So, we associate to \textcircled{H} a set of j different maps $\tilde{\epsilon}_j : \lambda_0 + X^\ast \rightarrow$ Virtual repns of $K_C^{\xi_j}$

Then $\sum \widetilde{C}_j(d_0 + d) \otimes F = \sum_{\substack{\mu \in \\ \text{weight} \\ \text{of } F}} \text{mult}_F(\mu) \widetilde{C}_j(d_0 + d + \mu)$

This is a hint for computing \widetilde{C}_j .

$$\begin{aligned} \text{This formula } &= (\dim \widetilde{C}_j(d_0 + d)) \cdot (\dim(F)) \\ &= \sum_{\substack{\mu \in \\ \text{weight} \\ \text{of } F}} \text{mult}_F(\mu) \dim \widetilde{C}_j(d_0 + d + \mu) \end{aligned}$$

Lemma: If $\varphi: d_0 + X^* \rightarrow \mathbb{C}$ satisfies

$$\varphi(d_0 + d) \dim(F) = \sum_{\mu \text{ weight}} \text{mult}_F(\mu) \varphi(d_0 + d + \mu),$$

then φ is a harmonic polynomial in the symmetric algebra of $h, S(h)$.